Constructions of Vertex Operator Algebras and Their Modules

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[1] Happy to hear your questions or comments!

Meeting the family

We have heard a lot about VOAs and their modules[2].

These are VOAs and their modules...

1. ... associated to the Virasoro algebra
2. ... associated to affine Lie algebras
3. ... associated to Heisenberg algebras

We will discuss [LL, 6.1-6.3] and relevant parts of other sections of the same book.

[2] [LL, FLM]

Outline

The conformal element a.k.a. Virasoro vector

- *Virasoro algebra* $\mathcal{L}$:= Lie algebra with basis $\{L_n\}_{n\in\mathbb{Z}} \cup \{\mathbf{c}\}$, with $\mathbf{c}$ central and with relations

$$[L_m, L_n] = (m-n)L_{m+n} + \tfrac{m^3-m}{12}\delta_{m,-n}\mathbf{c} \ .$$

- subalgebras: $\mathbb{C}L(-1) \oplus \mathbb{C}L(0) \oplus \mathbb{C}L(1) \cong \mathfrak{sl}_2$, $\mathbb{C}L_0 \oplus \mathbb{C}\mathbf{c}$ abelian, $\bigoplus_{n\geq 1} \mathbb{C}L_{\pm n}$

- V a VOA $\Rightarrow \exists$ *conformal element* ω, $Y(\omega, z) =: \sum_{n\in\mathbb{Z}} L(n) z^{-n-2}$, and *central charge* (*rank*) $\ell \in \mathbb{C}$ such that $L_n \mapsto L(n), \mathbf{c} \mapsto \ell \cdot \mathrm{id}_V$ is a representation of $\mathcal{L}$ on V.
  creation property $\Rightarrow L(n)\mathbf{1} = \omega_{n+1}\mathbf{1} = 0$ for $n \geq -1$.

Look for modules of the Virasoro algebra!

- $\mathcal{L}$ is a graded Lie algebra with $\mathcal{L}_{(0)} := \mathbb{C}L_0 \oplus \mathbb{C}\mathbf{c}$ and $\mathcal{L}_{(\pm n)} := \mathbb{C}L_{\mp n}$ for $n \geq 1$
- $\mathbb{C}_{\ell,h} :=$ one-dimensional $\mathcal{L}_{(\leq 0)}$-module, $\mathbf{c}$, L_0 act as $\ell, h \in \mathbb{C}$, L_n acts as 0 for $n \geq 1$
  $\Rightarrow \mathbb{C}_{\ell,0}$ is a $\mathcal{L}_{(\leq 1)}$-module with $\mathcal{L}_{(1)}$ acting as 0
- $\Rightarrow$ induced $\mathcal{L}$-modules

$$V_{Vir}(\ell, 0) := U(\mathcal{L}) \otimes_{U(\mathcal{L}_{(\leq 1)})} \mathbb{C}_{\ell,0}$$

$$M_{Vir}(\ell, h) := U(\mathcal{L}) \otimes_{U(\mathcal{L}_{(\leq 0)})} \mathbb{C}_{\ell,h}$$

with basis elements $L(-m_1) \cdots L(-m_r)\mathbf{1} = L_{-m_1} \cdots L_{-m_r} \otimes 1$ for $r \geq 0$, $m_1 \geq \cdots \geq m_r \geq 2$ or $\ldots \geq 1$, resp., where $L(n) :=$ operator for L_n, $\mathbf{1} := 1 \otimes 1$

Weight grading of $V_{Vir}(\ell, 0)$ and $M_{Vir}(\ell, h)$

$[L_0, L_{-m}] = mL_{-m} \Rightarrow$

$$L(0)L(-m_1)\cdots L(-m_r)\mathbf{1} = (m_1 + \cdots + m_r)L(-m_1)\cdots L(-m_r)\mathbf{1} .$$

$\Rightarrow$ for every $k \in \mathbb{Z}$, $\{L(-m_1)\cdots L(-m_r)\mathbf{1}\}_{m_1+\cdots+m_r=k}$ span the $L(0)$-eigenspace with eigenvalue k or $k + h$, respectively.
This yields a $\mathbb{Z}$- or $\mathbb{C}$-grading ("*by weight*").

We observe:

- each eigenspace is finite-dimensional
- for sufficiently negative eigenvalues, the eigenspaces are 0

(Recall: $r \geq 0, m_1 \geq \cdots \geq m_r \geq 2$ or $\ldots \geq 1$.)

$\Leftrightarrow$ "*two grading restrictions*"

Look (no) further for modules of the Virasoro algebra!

- *Universality of $V_{Vir}(\ell, 0)$*: For every $\mathcal{L}$-module M of central charge ℓ with $e \in M$ such that $L(n)e = 0$ for $n \geq -1$, there is a unique quotient map $V_{Vir}(\ell, 0) \to M$ sending $\mathbf{1} \mapsto e$.

  (Recall: This is the case if M is a VOA due to the creation property.)

- *Universality of $M_{Vir}(\ell, h)$*: For every $\mathcal{L}$-module M of central charge ℓ with $e \in M$ such that $L(0)e = he$ and $L(n)e = 0$ for $n \geq 1$, there is a unique quotient map $M_{Vir}(\ell, 0) \to M$ sending $\mathbf{1} \mapsto e$.

(Explicitly, these maps send $L(-m_1) \cdots L(-m_r)\mathbf{1} \mapsto L(-m_1) \cdots L(-m_r)e$.)

Recap ([LL, chapter 5])

W a vector space, $\mathcal{E}(W) := \mathrm{Hom}(W, W((x)))$ is a *weak VA* (i.e., without the Jacobi identity) with
$Y_{\mathcal{E}}(a(x), x_0)b(x) := \mathrm{Res}_{x_1}(x_0^{-1}\delta(\frac{x_1-x}{x_0})a(x_1)b(x) - x_0^{-1}\delta(\frac{-x+x_1}{x_0})b(x)a(x_1))$
or $a(x)_n b(x) := \mathrm{Res}_{x_1}((x_1 - x)^n a(x_1)b(x) - (-x + x_1)^n b(x)a(x_1))$
a, b *local* $:\Leftrightarrow (x_1 - x_2)^k [a(x_1), b(x_2)] = 0$ for $k \gg 0$

Theorem ([LL, 5.5.18])

$S \subset \mathcal{E}(W)$ a set of mutually local weak vertex operators $\Rightarrow \langle S \rangle$, the weak VA generated by S, is a VA and equals $\mathrm{span}\{a^{(1)}(x)_{n_1} \cdots a^{(r)}(x)_{n_r} 1_W\}$.

In our situation:
W a $\mathcal{L}$-module is called *restricted* $:\Leftrightarrow \forall w \in W, L_n w = 0$ for $n \gg 0$
$\Rightarrow L_W := \sum_{n\in\mathbb{Z}} L(n)x^{-n-2}$ lies in $\mathcal{E}(W)$

L_W is self-local (Check!) $\Rightarrow \langle L_W \rangle$ is a VA

Scratch-work: L_W is self-local ([LL])

W a restricted $\mathcal{L}$-module of central charge ℓ, $L_W(x) := \sum_n L_W(n)x^{-n-1}$.

$$x_2^{-1}\delta(\tfrac{x_1}{x_2}) = \sum_n x_1^n x_2^{-n-1} \qquad \partial_{x_1}^k x_2^{-1}\delta(\tfrac{x_1}{x_2}) = \sum_n n\cdots(n-k+1)x_1^{n-k}x_2^{-n-1}$$

$$\frac{(-1)^k}{k!}\partial_{x_1}^k x_2^{-1}\delta(\tfrac{x_1}{x_2}) = (x_1-x_2)^{k-1} - (-x_2+x_1)^{k-1}$$

$$[L_W(x_1), L_W(x_2)] = \sum_{m,n}[L_W(m)x_1^{-m-2}, L_W(n)x_2^{-n-2}]$$

$$= \sum_{m,n}((m-n)L(m+n) + \tfrac{\ell}{12}(m^3-m)\delta_{m,-n})x_1^{-m-2}x_2^{-n-2}$$

$$= \sum_{m,n}(((-m-n-2)+(2m+2))L(m+n) + \tfrac{\ell}{12}(m^3-m)\delta_{m,-n})x_1^{-m-2}x_2^{m+1}x_2^{-m-n-3}$$

$$= \sum_{m}(L_W'(x_2) + 2(m+1)L_W(x_2))x_1^{-m-2}x_2^{m+1} + \tfrac{\ell}{12}(m-1)m(m+1)x_1^{-m-2}x_2^{m-2}$$

$$= L_W'(x_2)x_2^{-1}\delta(\tfrac{x_1}{x_2}) - 2L_W(x_2)\partial_{x_1}x_2^{-1}\delta(\tfrac{x_1}{x_2}) - \tfrac{\ell}{12}\partial_{x_1}^3 x_2^{-1}\delta(\tfrac{x_1}{x_2})$$

$$\Rightarrow (x_1-x_2)^4[L_W(x_1), L_W(x_2)] = 0.$$

Scratch-work: L_W is self-local (variation)

f a polynomial, $a, b \in \mathbb{Z}$.

$$(x_1 - x_2) \sum_m f(m) x_1^{-m+a} x_2^{m+b} = \sum_m f(m)(x_1^{-m+a+1} x_2^{m+b} - x_1^{-m+a} x_2^{m+b+1})$$
$$= \sum_m \underbrace{(f(m+1) - f(m))}_{=:g(m)} x_1^{-m+a} x_2^{m+b+1} ,$$

but $\deg g < \deg f \Rightarrow (x_1 - x_2)^k \sum_m f(m) x_1^{-m+a} x_2^{m+b} = 0$ for $k > \deg f$. Now

$$[L_W(x_1), L_W(x_2)] = \sum_{m,n} [L_W(m) x_1^{-m-2}, L_W(n) x_2^{-n-2}]$$
$$= \sum_{m,n} ((m-n)L(m+n) + \tfrac{\ell}{12}(m^3 - m)\delta_{m,-n}) x_1^{-m-2} x_2^{-n-2} \qquad s := m+n$$
$$= \sum_{m,s} ((2m - s)L(s) + \tfrac{\ell}{12}(m^3 - m)\delta_{s,0}) x_1^{-m-2} x_2^{-s+m-2}$$
$$= \sum_m \left(2m(\sum_s L(s) x_2^{-s}) - (\sum_s sL(s) x_2^{-s}) + \tfrac{\ell}{12}(m^3 - m) \right) x_1^{-m-2} x_2^{m-2}$$

$\Rightarrow (x_1 - x_2)^4 [L_W(x_1), L_W(x_2)] = 0.$

$\mathcal{L}$-modules as VOAs

Theorem ([LL, 6.1.5])

V an $\mathcal{L}$-module of central charge $\ell \in \mathbb{C}$ generated by $\mathbf{1} \in V$ such that $L(n)\mathbf{1} = 0$ for $n \geq -1$ $\Rightarrow$ V is "naturally" a VOA.

VOA	$\mathcal{L}$-module
$\mathbf{1}$	$\mathbf{1}$
ω	$L(-2)\mathbf{1}$
$Y(L(n_1)\cdots L(n_r)\mathbf{1}, x)$	$L_V(x)_{n_1+1}\cdots L_V(x)_{n_r+1}\,\mathrm{id}_V$
$Y(\omega, x)$	$L_V(x)$

where $L_V(x) := \sum_{n\in\mathbb{Z}} L(n)x^{-n-2} \in \mathcal{E}(V)$

Recall [LL, 5.7.1+4]

V a vector space (restricted $\mathcal{L}$-module), $\mathbf{1} \in V$, $d \in \mathrm{End}(V)$,
$d(\mathbf{1}) = 0$, $T \subset V$,
$Y_0(\cdot, x) : T \to \mathcal{E}(V) = \mathrm{Hom}(V, V((x)))$, $a \mapsto \sum_{n\in\mathbb{Z}} a_n x^{-n-1}$,
V spanned by $\{a^{(1)}_{n_1} \cdots a^{(r)}_{n_r}\mathbf{1}\}$ for $r \geq 0$, $a^{(i)} \in T$, $n_i \in \mathbb{Z}$.

Extend Y_0 to V; $Y(a^{(1)}_{n_1} \cdots a^{(r)}_{n_r}\mathbf{1}, x) := a^{(1)}(x)_{n_1} \cdots a^{(r)}(x)_{n_r} 1_V$.

This yields VOA (with $d(v) = v_{-2}\mathbf{1}$, $\omega := L(-2)\mathbf{1}$) if

- vacuum + creation property hold for $T, Y_0, \mathbf{1}$
- $Y_0(a, x)$, $Y_0(b, x)$ for $a, b \in T$ are mutually local
- $[d, Y_0(a, x)] = \frac{d}{dx} Y_0(a, x)$ for $a \in T$
- $L(-1) = d$ (as endomorphisms of V)
- $\omega \in T$, $Y_0(\omega, x) = \sum_n L(n) x^{-n-2}$ $(= L_V(x))$
- $\forall a \in T\, \exists m \in \mathbb{Z} : [L(0), a(x)] = m a(x) + x \frac{d}{dx} a(x)$
- the two grading restrictions hold

$\mathcal{L}$-modules as VOAs

Theorem ([LL, 6.1.5])

V an $\mathcal{L}$-module of central charge $\ell \in \mathbb{C}$ generated by $\mathbf{1} \in V$ such that $L(n)\mathbf{1} = 0$ for $n \geq -1$ $\quad \Rightarrow V$ is "naturally" a VOA.

Proof:

- By the universal property of $V_{Vir}(\ell, 0)$, V is a quotient of $V_{Vir}(\ell, 0)$, in particular, a restricted $\mathcal{L}$-module.
- Let $d := L(-1)$, $\omega := L(-2)\mathbf{1}$,
  $T := \{\omega\} \subset V$, $Y_0(\omega, x) := L_V(x)$.
  Then this extends to a VOA structure without the two grading restrictions by [LL, 5.7.4] if L_V is self-local,
  $[L(-1), L_V(x)] = \frac{d}{dx} L_V(x)$ and
  $[L(0), L_V(x)] = 2L_V(x) + x\frac{d}{dx} L_V(x)$. (Check!)
- Again as V is a quotient of $V_{Vir}(\ell, 0)$, we get the two grading restrictions.

Recap [LL, 5.7.6]

V a VA generated by a local subset T, W a vector space, $Y_W^0(\cdot, x) = \iota_W^0 : T \to \mathcal{E}(W), a \mapsto a_W(x)$ can be extended to map $Y_W(\cdot, x) = \iota_W : V \to \mathcal{E}(W)$ making W a V-module if

- $\iota_W(\mathbf{1}) = \mathrm{id}_W$
- $\iota_W(a_n v) = a_W(x)_n \iota_W(v)$ for all $a \in T, v \in V$

$\mathcal{L}$-modules as VA modules

Theorem ([LL, 6.1.7])

W a restricted $\mathcal{L}$-module of central charge $\ell \in \mathbb{C}$
$\Rightarrow$ W is "naturally" a VA module of $V_{Vir}(\ell, 0)$.

VOA module	$\mathcal{L}$-module
$Y_W(L(n_1)\cdots L(n_r)\mathbf{1}, x)$	$L_W(x)_{n_1+1}\cdots L_W(x)_{n_r+1}\,\mathrm{id}_W$
$Y_W(\omega, x)$	$L_W(x)$

Proof:
$U := \mathrm{span}\{L_W(x)_{n_1}\cdots L_W(x)_{n_r}1_W : r \geq 0, n_i \in \mathbb{Z}\} \subset \mathcal{E}(W)$
$\Rightarrow$ U is an $\mathcal{L}$-module with L_n acting as $L_W(x)_{n+1}$, $L_W(x)_n 1_W = 0$ for $n \geq 0$
universality of $V := V_{Vir}(\ell, 0) \Rightarrow \exists\, \psi : V \to U$ $\mathcal{L}$-module map such that $\mathbf{1} \mapsto \mathrm{id}_W$, $\psi(\omega) = L_W$. $\Rightarrow \psi(\omega_n v) = L_W(x)_n \psi(v)$
$T := \{\omega\} \overset{\text{LL, 5.7.6}}{\Rightarrow} W$ a V-module

$\mathcal{L}$-modules as VOA modules

Theorem ([LL, 6.1.8])

Restricted $\mathcal{L}$-modules of central charge ℓ are just the VA modules of $V_{Vir}(\ell, 0)$, in the respective "natural" interpretations.
Under this correspondence, VOA modules of $V_{Vir}(\ell, 0)$ correspond to restricted $\mathcal{L}$-modules which

- *are graded by $L(0)$-eigenvalues and*
- *have the two grading restrictions.*

$M := M_{Vir}(\ell, h)$.
$\Rightarrow$ M is a VOA module of $V_{Vir}(\ell, 0)$
$T :=$ sum of proper submodules of M.
$\Rightarrow$ $L := L_{Vir}(\ell, h) := M/T$ is the unique irreducible quotient
$\Rightarrow$ L is an irreducible VOA module of $V_{Vir}(\ell, 0)$ – and those are all!

Irreducible VOA modules of $V_{Vir}(\ell, 0)$

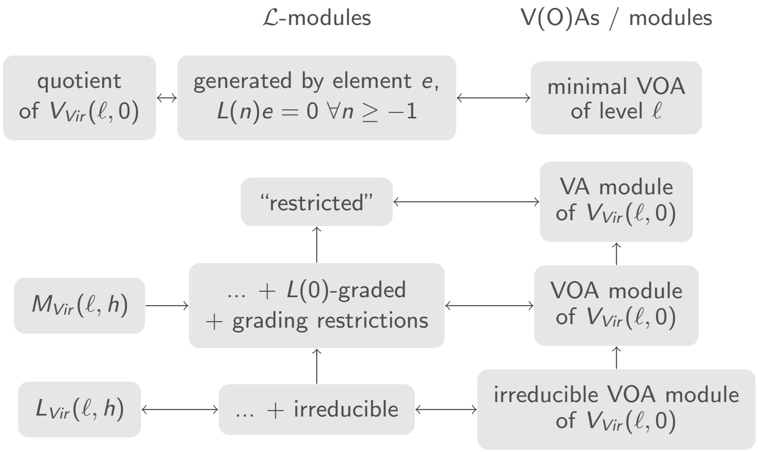
Theorem ([LL, 6.1.12])

The irreducible VOA modules of $V_{Vir}(\ell, 0)$ are just the $L_{Vir}(\ell, h)$ for all h.

Proof: W any irreducible VOA module of $V_{Vir}(\ell, 0)$.

- Pick a non-zero element w in the lowest $L(0)$-weight space, set $h :=$ the weight of w.
- M, T, L as above. Universal property of $M \Rightarrow \mathcal{L}$-module map $M \to W$ sending $\mathbf{1}$ to w.
- W irreducible $\Rightarrow$ map is onto and has kernel T.
- $W \cong L$ as $\mathcal{L}$-modules, and hence as VOA modules.

Summary



(arrows mean "special case of")

Outline

VOAs and modules

(Untwisted) Affine Lie algebras

- $\mathfrak{g}$ a Lie algebra + invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$
We define the affine Lie algebra $\hat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k}$ with $\mathbf{k}$ central and with

$$[a \otimes t^m, b \otimes t^n] := [a, b] \otimes t^{m+n} + m\langle a, b\rangle \delta_{m,-n}\mathbf{k} \ .$$

- $\hat{\mathfrak{g}}_{(0)} := \mathfrak{g} \oplus \mathbb{C}\mathbf{k}$, $\hat{\mathfrak{g}}_{(n)} := \mathfrak{g} \otimes t^{-n}$ for $n \neq 0$
$\Rightarrow \hat{\mathfrak{g}}_{(\pm)}, \hat{\mathfrak{g}}_{(\leq 0)}$ subalgebras

Locality of vertex operators

For $a \in \mathfrak{g}$, write $a(n) := a \otimes t^n \in \hat{\mathfrak{g}}$
and $a(x) := \sum_{n \in \mathbb{Z}} a(n) x^{-n-1} \in \hat{\mathfrak{g}}[[x, x^{-1}]]$.

$$\Rightarrow [a(x_1), b(x_2)] = [a, b](x_2)\, x_2^{-1} \delta\left(\frac{x_1}{x_2}\right) - \langle a, b\rangle\, \partial_{x_1} x_2^{-1} \delta\left(\frac{x_1}{x_2}\right) \mathbf{k}$$

$$\Rightarrow (x_1 - x_2)^2 [a(x_1), b(x_2)] = 0$$

$\Rightarrow$ For any restricted ($\forall a, w : a(n)w = 0$ if $n \gg 0$) $\hat{\mathfrak{g}}$-module W,
$S := \{a(x)\}_{a \in \mathfrak{g}} \subset \mathcal{E}(W)$ is local

$\overset{\text{LL, 5.5.18}}{\Rightarrow}$ $\langle S \rangle$ is a VA spanned by $\{a^{(1)}(x)_{n_1} \cdots a^{(r)}(x)_{n_r} 1_W\}$.

Scratch-work: Locality of vertex operators

Recall: f a polynomial, $a, b \in \mathbb{Z}$
$\Rightarrow (x_1 - x_2)^{1+\deg f} \sum_m f(m) x_1^{-m+a} x_2^{m+b} = 0.$

Now

$$
\begin{aligned}
[a(x_1), b(x_2)] &= \sum_{m,n} [a(m), b(n)] x_1^{-m-1} x_2^{-n-1} \\
&= \sum_{m,n} ([a,b](m+n) + m\langle a,b\rangle \delta_{m,-n} \mathbf{k}) x_1^{-m-1} x_2^{-n-1} \qquad s := m+n \\
&= \sum_{m,s} ([a,b](s) + m\langle a,b\rangle \delta_{s,0} \mathbf{k}) x_1^{-m-1} x_2^{-s+m-1} \\
&= \sum_m \left((\sum_s [a,b](s) x_2^{-s}) + m\langle a,b\rangle \mathbf{k} \right) x_1^{-m-1} x_2^{m-1}
\end{aligned}
$$

$\Rightarrow (x_1 - x_2)^2 [a(x_1), b(x_2)] = 0.$

Recall [LL, 5.7.1+4]

V a vector space (restricted $\mathcal{L}$-module), $\mathbf{1} \in V$, $d \in \mathrm{End}(V)$, $d(\mathbf{1}) = 0$, $T \subset V$,
$Y_0(\cdot, x) : T \to \mathcal{E}(V) = \mathrm{Hom}(V, V((x)))$, $a \mapsto \sum_{n\in\mathbb{Z}} a_n x^{-n-1}$,
V spanned by $\{a^{(1)}_{n_1} \cdots a^{(r)}_{n_r} \mathbf{1}\}$ for $r \geq 0$, $a^{(i)} \in T, n_i \in \mathbb{Z}$.

Extend Y_0 to V; $Y(a^{(1)}_{n_1} \cdots a^{(r)}_{n_r} \mathbf{1}, x) := a^{(1)}(x)_{n_1} \cdots a^{(r)}(x)_{n_r} 1_V$.

This yields VOA (with $d(v) = v_{-2}\mathbf{1}$, $\omega := L(-2)\mathbf{1}$) if

- vacuum + creation property hold for $T, Y_0, \mathbf{1}$
- $Y_0(a, x)$, $Y_0(b, x)$ for $a, b \in T$ are mutually local
- $[d, Y_0(a, x)] = \frac{d}{dx} Y_0(a, x)$ for $a \in T$
- $L(-1) = d$ (as endomorphisms of V)
- $\omega \in T$, $Y_0(\omega, x) = \sum_n L(n) x^{-n-2}$ $(= L_V(x))$
- $\forall a \in T \, \exists m \in \mathbb{Z} : [L(0), a(x)] = m a(x) + x \frac{d}{dx} a(x)$
- the two grading restrictions hold

A vertex algebra for every level

$\mathbb{C}_\ell$:= $\hat{\mathfrak{g}}_{(\leq 0)}$-module such that $\mathbf{k}$ acts as ℓ, everything else as 0
$V_{\hat{\mathfrak{g}}}(\ell, 0) := \mathrm{Ind}_{\hat{\mathfrak{g}}_{(\leq 0)}}^{\hat{\mathfrak{g}}} \mathbb{C}_\ell = U(\hat{\mathfrak{g}}) \otimes_{U(\hat{\mathfrak{g}}_{(\leq 0)})} \mathbb{C}_\ell$
$\Rightarrow V_{\hat{\mathfrak{g}}}(\ell, 0)$ spanned by $\{a^{(1)}(-m_1) \cdots a^{(r)}(-m_r)\mathbf{1}\}$

$V_{\hat{\mathfrak{g}}}(\ell, 0)_{(n)} := \mathrm{span}\{a^{(1)}(-m_1) \cdots a^{(r)}(-m_r)\mathbf{1} : m_1 + \cdots + m_r = n\}$

d derivation on $V_{\hat{\mathfrak{g}}}(0, \ell)$ defined by $\mathbf{k} \mapsto 0$, $a(n) \mapsto -na(n-1)$
$\Rightarrow [d, a(n)] = -na(n-1)$ as operators on $V_{\hat{\mathfrak{g}}}(\ell, 0)$

Theorem ([LL, 6.2.11])

There is a unique structure map Y such that $(V_{\hat{\mathfrak{g}}}(\ell, 0), Y, 1)$ is a VA and such that $Y(a, x) = a(x)$ for all $a \in \mathfrak{g}$.
Explicitly, $Y(a^{(1)}(n_1) \cdots a^{(r)}(n_r)\mathbf{1}, x) = a^{(1)}(x)_{n_1} \cdots a^{(r)}(x)_{n_r} 1$.

$V_{\hat{\mathfrak{g}}}(\ell, 0)$-modules = restricted $\hat{\mathfrak{g}}$-modules of level ℓ

Theorem ([LL, 6.2.13])

W a VA module of $V_{\hat{\mathfrak{g}}}(\ell, 0)$
$\Rightarrow$ W a restricted $\hat{\mathfrak{g}}$-module of level ℓ with $a_W(x) = Y_W(a, x)$.

W a restricted $\hat{\mathfrak{g}}$-module of level ℓ
$\Rightarrow$ W a VA module of $V_{\hat{\mathfrak{g}}}(\ell, 0)$ with

$$Y_W(a^{(1)}(n_1) \cdots a^{(r)}(n_r)\mathbf{1}, x) = a_W^{(1)}(x)_{n_1} \cdots a_W^{(r)}(x)_{n_r} 1_W \ .$$

Now for VOAs

Assume $d := \dim \mathfrak{g} < \infty$, $\langle \cdot, \cdot \rangle$ non-degenerate.
Pick an orthonormal basis $(u^{(i)})_i$.
$\Rightarrow$ Casimir element $\Omega := \sum_i u^{(i)} u^{(i)} \in U(\mathfrak{g})$ is central and independent of the choice of $(u^{(i)})_i$.

Assume Ω acts as $2h \in \mathbb{C}$ on $\mathfrak{g}$ (under the adjoint action), $\ell \neq -h$.

$$\omega := \frac{1}{2(\ell+h)} \sum_{i=1}^{d} u^{(i)}(-1) u^{(i)}(-1)\mathbf{1} \quad \in V_{\hat{\mathfrak{g}}}(\ell, 0)_{(2)}$$

Theorem ([LL, 6.2.15])

The components $L(n)$ of $Y(\omega, x)$ viewed as operators on any restricted $\hat{\mathfrak{g}}$-module of level ℓ satisfy the Virasoro relations corresponding to the central charge $d\ell/(\ell + h)$.
Furthermore, $L(0)v = nv$ for all $v \in V_{\hat{\mathfrak{g}}}(\ell, 0)_{(n)}$ and $L(-1) = \mathcal{D}$ on $V_{\hat{\mathfrak{g}}}(\ell, 0)$.

Now(!) for VOAs

...} $\Rightarrow$

Theorem ([LL, 6.2.18])

If $\mathfrak{g}$ is a d-dimensional Lie algebra with non-degenerate symmetric bilinear form such that Ω acts on $\mathfrak{g}$ as scalar $2h$ and $\ell \neq -h$, then $V_{\hat{\mathfrak{g}}}(\ell, 0)$ is a VOA of central charge $d\ell/(\ell + h)$ with conformal element ω as above.
Furthermore, $L(0)$-eigenvalues are determined by the chosen $\mathbb{Z}$-grading and $\mathfrak{g} = V_{\hat{\mathfrak{g}}}(\ell, 0)_{(1)}$ generates $V_{\hat{\mathfrak{g}}}(\ell, 0)$ as VA.

Irreducible modules of $V_{\hat{\mathfrak{g}}}(\ell, 0)$

U a finite-dimensional $\mathfrak{g}$-module such that Ω acts as $h_U \in \mathbb{C}$
$\Rightarrow$ U a $\hat{\mathfrak{g}}_{(\leq 0)}$-module where $\mathbf{k}$ acts as ℓ and $\hat{\mathfrak{g}}_{(-)}$ acts as 0

$W := \mathrm{Ind}_{\mathfrak{g}}^{\hat{\mathfrak{g}}}(U) = U(\hat{\mathfrak{g}}) \otimes_{\hat{\mathfrak{g}}_{(\leq 0)}} U$ is a $\hat{\mathfrak{g}}$-module
$L_{\hat{\mathfrak{g}}}(\ell, U) :=$ the unique irreducible quotient of W

Theorem ([LL, 6.2.21])

W is a VOA module of $V_{\hat{\mathfrak{g}}}(\ell, 0)$.

Theorem ([LL, 6.2.23])

The irreducible VOA modules of $V_{\hat{\mathfrak{g}}}(\ell, 0)$ are just the modules $L_{\hat{\mathfrak{g}}}(\ell, U)$ for all finite-dimensional irreducible $\mathfrak{g}$-modules U.

$\Rightarrow$ [LL, 6.2.25]: $L_{\hat{\mathfrak{g}}}(\ell, 0) := L_{\hat{\mathfrak{g}}}(\ell, \mathbb{C})$ is a simple VOA.

Outline

VOAs and modules

From affine Lie algebras to Heisenberg algebras

Specialize $\mathfrak{g}$ to be a commutative Lie algebra and call it $\mathfrak{h}$.
$\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{k}$, $\mathbf{k}$ central,

$$[\alpha \otimes t^m, \beta \otimes t^n] = \langle \alpha, \beta \rangle m \delta_{m,-n} \mathbf{k} \ .$$

$\hat{\mathfrak{h}}_{(0)} = \mathfrak{h} \oplus \mathbb{C}\mathbf{k}$, $\hat{\mathfrak{h}}_{(n)} = \mathfrak{h} \otimes t^{-n}$,
$\hat{\mathfrak{h}} = \mathfrak{h} \oplus \hat{\mathfrak{h}}_*$ with $\hat{\mathfrak{h}}_* := \hat{\mathfrak{h}}_{(-)} \oplus \mathbb{C}\mathbf{k} \oplus \hat{\mathfrak{h}}_{(+)}$, a *Heisenberg algebra* (i.e. it has a one-dimensional center which equals the commutator subalgebra).

For any $\hat{\mathfrak{h}}$-module W, the action of $\alpha \otimes t^n$ is denoted by $\alpha(n)$, and $\alpha_W(x) := \sum_n \alpha(n) x^{-n-1}$.

Action of the Casimir

Identify $\mathfrak{h}$ with $\mathfrak{h}^*$ using $\langle\cdot,\cdot\rangle$.
For any $\alpha \in \mathfrak{h}$, let $\mathbb{C}_\alpha$ be the one-dimensional $\mathfrak{h}$-module with $\beta \in \mathfrak{h}$ acting as $\langle\beta,\alpha\rangle$. Then

$$\Omega \cdot 1 = \sum_i u^{(i)} u^{(i)} \cdot 1 = \sum_i \langle u^{(i)}, \alpha\rangle^2 1 = \langle\alpha,\alpha\rangle 1 \ .$$

Regard $\mathbb{C}_\alpha$ as $\hat{\mathfrak{h}}_{(\leq 0)}$-module such that **k** acts as ℓ, $\hat{\mathfrak{h}}_{(-)}$ acts as 0.
$M(\ell,\alpha) := \mathrm{Ind}_{\mathfrak{h}}^{\hat{\mathfrak{h}}}(\mathbb{C}_\alpha) = U(\hat{\mathfrak{h}}) \otimes_{U(\hat{\mathfrak{h}}_{(\leq 0)})} \mathbb{C}_\alpha$
$\Rightarrow M(\ell,0) = V_{\hat{\mathfrak{h}}}(\ell,0)$, Ω acts as 0 (“$= 2h$”).

Theorem ([LL, 6.3.2+3])

For $\ell \neq 0$, $V_{\hat{\mathfrak{h}}}(\ell,0) = M(\ell,0)$ is naturally a VOA.
For any $\alpha \in \mathfrak{h}$, $M(\ell,\alpha)$ is naturally a module of $V_{\hat{\mathfrak{h}}}(\ell,0)$.

A realization of $\hat{\mathfrak{h}}$ and $\hat{\mathfrak{h}}_*$

$d := \dim \mathfrak{h}$, $(u^{(i)})_i$ an orthonormal basis of $\mathfrak{h}$,
$P(\ell, \alpha) := \mathbb{C}[x_{ij}]_{1 \leq i,j \leq d}$.
Define an action of $\hat{\mathfrak{h}}$ on $P(\ell, \alpha)$: for $n > 0$,

- $\mathbf{k}$ acts as ℓ
- $u^{(i)}(0) := \langle u^{(i)}, \alpha \rangle$
- $u^{(i)}(n) := n\ell \frac{d}{dx_{in}}$
- $u^{(i)}(-n) := x_{in}$ (left-multiplication in $P(\ell, \alpha)$.

Theorem ([LL, 6.3.4])

This makes $P(\ell, \alpha)$ an irreducible $\hat{\mathfrak{h}}$-module and and irreducible $\hat{\mathfrak{h}}_*$-module.

$M(\ell, \alpha)$ and $P(\ell, \alpha)$

W an $\hat{\mathfrak{h}}_*$-module, $w \in W$ is called *vacuum vector* if $\hat{\mathfrak{h}}_{(-)} w = 0$.

Theorem ([LL, 6.3.8])

– *$M(\ell, \alpha)$ is an irreducible $\hat{\mathfrak{h}}$-module / $\hat{\mathfrak{h}}_*$-module.*
– *Any $\hat{\mathfrak{h}}$-submodule generated by a vacuum vector $\cong M(\ell, \alpha)$.*
– *$M(\ell, 0)$ is the unique irreducible $\hat{\mathfrak{h}}_*$-module containing a vacuum vector.*

Proof:

- $\mathbb{C} \subset P(\ell, \alpha)$ is equivalent to $\mathbb{C}_\alpha$
  universality of $M(\ell, \alpha) \Rightarrow \exists$ module map $M(\ell, \alpha) \to P(\ell, \alpha)$
  irreducibility of $P(\ell, \alpha) \Rightarrow M(\ell, \alpha) \cong P(\ell, \alpha)$ as $\hat{\mathfrak{h}}$-modules
- W an $\hat{\mathfrak{h}}_*$-module of level ℓ generated by a vacuum vector w
  $\Rightarrow W$ an $\hat{\mathfrak{h}}$-module with $\mathfrak{h}$ acting as 0
  $\Rightarrow \mathbb{C}w \cong \mathbb{C}_\alpha$ for $\alpha = 0$, $W \cong M(\ell, 0) \cong P(\ell, 0)$

VOAs and modules, revisited

As a consequence, we get the following improvements:

Theorem ([LL, 6.3.9])

For $\ell \neq 0$, $V_{\hat{\mathfrak{h}}}(\ell, 0) = M(\ell, 0)$ is a simple VOA.
For any $\alpha \in \mathfrak{h}$, $M(\ell, \alpha)$ is one of its irreducible modules, and we obtain all irreducible modules in this way.

Theorem ([LL, 6.3.10])

For $\ell \neq 0$, $V_{\hat{\mathfrak{h}}}(\ell, 0) \cong V_{\hat{\mathfrak{h}}}(1, 0)$ as VOAs.

Proof: $V_{\hat{\mathfrak{h}}}(\ell, 0) \to V_{\hat{\mathfrak{h}}}(1, 0)$,

$$\alpha^{(1)}(n_1) \cdots \alpha^{(r)}(n_r)\mathbf{1} \mapsto (\sqrt{\ell})^r \alpha^{(1)}(n_1) \cdots \alpha^{(r)}(n_r)\mathbf{1} \ ,$$

for any choice of $\sqrt{\ell}$ is an isomorphism. (Check!)

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