Quantum Schur–Weyl duality and link invariants

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[1] Happy to hear your questions or comments!

Outline

Representation categories of Lie algebras

$\mathfrak{g}$: a Lie algebra over $\mathbb{F}$
Rep: the category of (finite dimensional) representations

- $\mathbb{F} \in \mathrm{Rep}$:

$$xr = 0 \qquad \forall r \in \mathbb{F}, x \in \mathfrak{g} \ .$$

- $M, N \in \mathrm{Rep} \Rightarrow M \otimes N \in \mathrm{Rep}$:

$$x(m \otimes n) = xm \otimes n + m \otimes xn \qquad \forall m \in M, n \in N \ .$$

- $M \in \mathrm{Rep} \Rightarrow M^* = \mathrm{Hom}_{\mathbb{F}}(M, \mathbb{F}) \in \mathrm{Rep}$:

$$xf = -f(x\cdot) \qquad \forall f \in M^* \ .$$

"$-$" implies:
$M \otimes M^* \to \mathbb{F}$ is a morphism in Rep, i.e., a $\mathfrak{g}$-module map

Associative algebras are nicer than Lie algebras...

$\mathcal{U} = \mathcal{U}(\mathfrak{g})$: the universal enveloping algebra of $\mathfrak{g}$ (associative + 1),
$\mathcal{U} = T(\mathfrak{g})/(xy - yx - [x,y])$; generated by $\{x\}_{x\in\mathfrak{g}}$ with relations

$$xy - yx = [x,y] \qquad \forall x, y \in \mathfrak{g} \; .$$

$\Rightarrow$ Rep = the category of (finite-dimensional) $\mathcal{U}$-modules

We have algebra maps $\varepsilon : \mathcal{U} \to \mathbb{F}$, $\Delta : \mathcal{U} \to \mathcal{U} \otimes \mathcal{U}$, $S : \mathcal{U} \to \mathcal{U}$ defined on the generators $\{x\}_{x\in\mathfrak{g}}$ by

$$\varepsilon(x) = 0, \qquad \Delta(x) = x \otimes 1 + 1 \otimes x, \qquad S(x) = -x$$

such that:

$$\mathbb{F} \in \text{Rep} : ur = \varepsilon(u)r \qquad \forall u \in U$$

$$M \otimes N \in \text{Rep} : u(m \otimes n) = \Delta(u)(m \otimes n)$$

$$M^* \in \text{Rep} : uf = f(S(u)\cdot)$$

… as long as they are Hopf algebras!

Algebras with additional structure maps ε, Δ, S as above satisfying certain axioms are called Hopf algebras.
Their representation categories are rigid monoidal categories ("they have duals and tensor products").

$\Rightarrow \mathcal{U}$ is a Hopf algebra

$\forall$ vector spaces V, W: $\tau_{V,W} : V \otimes W \to W \otimes V, v \otimes w \mapsto w \otimes v$.
$\Rightarrow \Delta = \tau_{\mathcal{U},\mathcal{U}} \circ \Delta$ ($\Leftrightarrow$: $\mathcal{U}$ is cocommutative)

$\Leftrightarrow$ Rep is symmetric monoidal with the symmetric braiding τ:
$\forall\ M, N \in \text{Rep}$, $\tau_{M,N}$ gives an isomorphism in Rep and $\tau^2 = \text{id}$.

Caution: Cocommutative Hopf algebras / symmetric monoidal categories are a special case!

Endomorphisms of tensor powers

We fix a module $M \in \mathrm{Rep}$ and $n \geq 1$.

$\Rightarrow M^{\otimes n} \in \mathrm{Rep}$ and we have an algebra map
$\phi(= \Delta^{n-1})\colon \mathcal{U} \to \mathrm{End}(M^{\otimes n})$

$s_i := \mathrm{id}^{\otimes(i-1)} \otimes \tau_{M,M} \otimes \mathrm{id}^{\otimes(n-i-1)}\colon M^{\otimes n} \to M^{\otimes n}$ for $1 \leq i < n$

- $\psi\colon S_n \to \mathrm{GL}(M^{\otimes n}), (i\ \ i+1) \mapsto s_i$ defines a group homo. ($s_i^2 = \mathrm{id}$, $s_i s_j s_i = s_j s_i s_j$ if $|i-j| = 1$, $s_i s_j = s_j s_i$ if $|i-j| > 1$)
- So we have an algebra map $\psi\colon \mathbb{F}[S_n] \to \mathrm{End}(M^{\otimes n})$.[2]
- For all $u \in \mathcal{U}$ and all i: $\phi(u), \psi(s_i)$ commute!

$\Rightarrow \phi(\mathcal{U}), \psi(\mathbb{F}[S_n])$ are commuting algebras in $\mathrm{End}(M^{\otimes n})$

[2] For a group G, $\mathbb{F}[G]$ is the algebra gen. by $\{e_g\}_{g \in G}$ with rel.s $e_g e_h = e_{gh}$.

Let's specialize to $\mathfrak{gl}_d(\mathbb{C})$

Let us specialize $\mathbb{F} = \mathbb{C}$, $\mathfrak{g} = \mathfrak{gl}_d(\mathbb{C})$, $M = \mathbb{C}^d$ for $d \geq 1$.

Schur–Weyl duality

$\phi(\mathcal{U}(\mathfrak{gl}_d)), \psi(\mathbb{C}[S_n])$ are (full!) commutators of each other in $\mathrm{End}((\mathbb{C}^d)^{\otimes n})$.

As a corollary, $(\mathbb{C}^d)^{\otimes n} = \bigoplus_\lambda V_\lambda \otimes W_\lambda$ for pairwise non-isomorphic irreducible $\mathfrak{gl}_d$-modules V_λ / S_n-modules W_λ.

More concretely, $\{\lambda\}$ can be taken to be the set of partitions of n with at most d parts. (Equivalently, partitions of n with all parts being at most d.)

Quantization

For $q \in \mathbb{C} \setminus \{0, 1\}$, $\mathcal{U}_q = \mathcal{U}_q(\mathfrak{gl}_d)$ is a Hopf algebra deformation of $\mathcal{U} = \mathcal{U}(\mathfrak{gl}_d)$ such that "$\mathcal{U}_q \to \mathcal{U}$ as $q \to 1$".

$\mathcal{U}_q$ (still) has $\mathbb{C}^d$ as a natural standard module.

The representation category Rep_q is rigid monoidal, but not symmetric anymore. S_n does not act on $(\mathbb{C}^d)^{\otimes n}$.

There is still a braiding $c_{M,N} \colon M \otimes N \to N \otimes M$ for $M, N \in \mathrm{Rep}_q$ with $c^2 \neq \mathrm{id}$ generally.

Rep_q is (still) a ribbon category: it has tensor products, duals, a braiding and twists, and they are compatible.

Quantum Schur–Weyl duality

$$S_n = \text{group generated by } s_1, \dots, s_{n-1} \text{ and relations:}$$
$$s_i^2 = 1, \quad \underbrace{s_i s_j s_i = s_j s_i s_j \text{ if } |i-j| = 1, \quad s_i s_j = s_j s_i \text{ if } |i-j| > 1}_{\text{braid relations}}$$

$$\mathcal{U}(\mathfrak{gl}_d) \xrightarrow{\ \phi\ } \mathrm{End}((\mathbb{C}^d)^{\otimes n}) \xleftarrow{\ \psi\ } \mathbb{C}[S_n]$$ double centralizer

braid group Br_n = group generated by $\sigma_1, \dots, \sigma_{n-1}$ with braid relations

Hecke algebra $\mathcal{H}_{q,n}$ = $\mathbb{C}$-algebra generated by $T_1, \dots, T_{n-1}$ with braid relations and $(T_i + q)(T_i - q^{-1}) = 1$

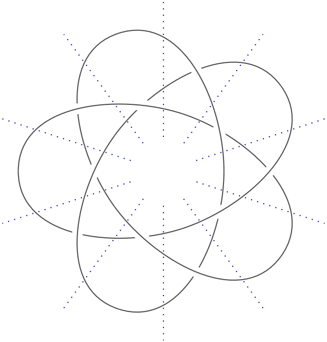
$$\mathbb{C}[\mathrm{Br}_n] \xrightarrow{\ \sigma_i \mapsto \mathrm{id}^{\otimes(i-1)} \otimes c \otimes \mathrm{id}^{\otimes \cdots}\ } \mathrm{End}((\mathbb{C}^d)^{\otimes n}), \qquad \mathbb{C}[\mathrm{Br}_n] \xrightarrow{\ \sigma_i \mapsto T_i\ } \mathcal{H}_{q,n}$$

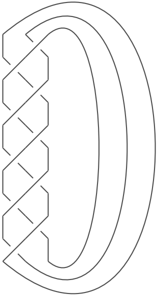
$$\mathcal{U}_q(\mathfrak{gl}_d) \longrightarrow \mathrm{End}((\mathbb{C}^d)^{\otimes n}) \longleftarrow \mathcal{H}_{q,n}$$ double centralizer

Outline

Braids and links – Alexander

(oriented) link := finite collection of smoothly embedded (oriented) circles in 3-space



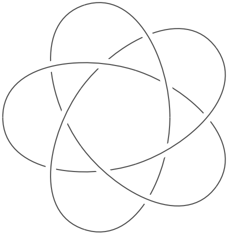


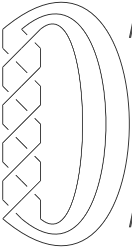
Alexander's theorem

(Oriented) links are closures of (oriented) braids.

Link invariants from quantum groups

We fix $M \in \mathrm{Rep}_q$.





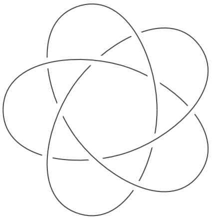
$$\mathbb{C} \searrow M^{\otimes 3} \otimes (M^*)^{\otimes 3} \xrightarrow{(\mathrm{id} \otimes c) \otimes \mathrm{id}^{\otimes 3},\ (c \otimes \mathrm{id}) \otimes \mathrm{id}^{\otimes 3},\ \dots,\ (c \otimes \mathrm{id}) \otimes \mathrm{id}^{\otimes 3}} M^{\otimes 3} \otimes (M^*)^{\otimes 3} \swarrow \mathbb{C}$$

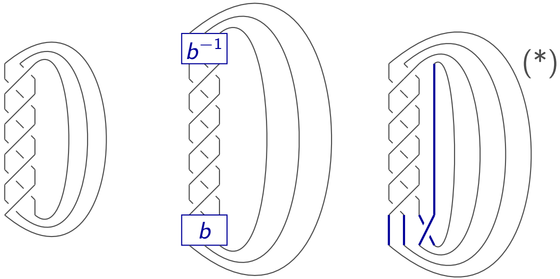
- $\mathbb{C}[\mathrm{Br}_n] \to \mathrm{End}(M^{\otimes n})$, braid $\mapsto$ endomorphism
- closing the braid $\leftrightarrow$ taking the trace

Reshetikhin–Turaev

The ribbon category Rep_q yields link invariants in this way.

Braids and links – Markov





Markov

$$\frac{\{\text{links}\}}{\text{isotopy}} \leftrightarrow \frac{\{\text{braids}\}}{\text{conjugations, Markov moves (*)}}$$

Link invariants from Hecke algebras

knots $\to$ braids $\to \bigcup_{n\geq 1} \mathbb{C}[\mathrm{Br}_n] \to \mathcal{H}_q := \bigcup_{n\geq 1} \mathcal{H}_{q,n}$

A linear map $\mathrm{Tr} : \mathcal{H}_q \to \mathbb{C}$ is called normalized Markov trace with parameter $z \in \mathbb{C}$
$:\Leftrightarrow \mathrm{Tr}(1) = 1, \quad \mathrm{Tr}(ab) = \mathrm{Tr}(ba), \quad \mathrm{Tr}(M(b)) = z\,\mathrm{Tr}(b)$
for all $a, b \in \mathcal{H}_q$, where $M(b)$ is the modification of b according to the Markov move.

Ocneanu

For all q, z, there is a unique normalized Markov trace.

Jones

Every normalized Markov trace yields an invariant for oriented links. Ocneanu's trace yields the two-parameter HOMFLYPT polynomial.

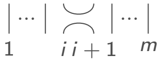
Outline

The Temperley–Lieb algebra

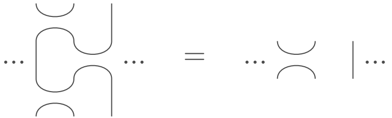
For $d = 2$, the image of $\mathbb{C}[\mathrm{Br}_n] \to \mathrm{End}((\mathbb{C}^2)^{\otimes n})$ is...

Temperley–Lieb algebra $\mathrm{TL}_n(\delta)$ generated by $u_1, \ldots, u_{n-1}$ with the relations:

$u_i^2 = \delta u_i$, $\quad u_i u_j u_i = u_i$ if $|i - j| = 1$, $\quad u_i u_j = u_j u_i$ if $|i - j| > 1$.

Graphically, u_i corresponds to  and composition corresponds to stacking diagrams ("crossingless matchings"), where circles are evaluated to δ.

E.g.,

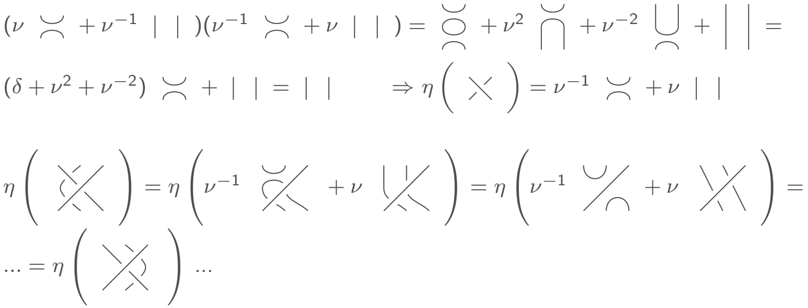
$u_i u_{i+1} u_i =$  $= u_i$

Braids and the Temperley–Lieb algebra

For any $\nu \in \mathbb{C}$, we have a group homomorphism $\eta\colon \mathrm{Br}_n \to \mathrm{TL}_n(\delta)$ sending $\sigma_i \mapsto \nu u_i + \nu^{-1}$, i.e.,

for $\delta = -\nu^2 - \nu^{-2}$.

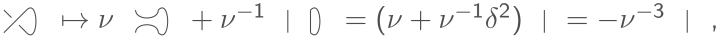
Pf.: By graphical calculus, e.g.,

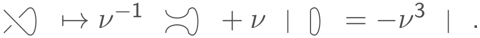


Always trouble with the Markov move

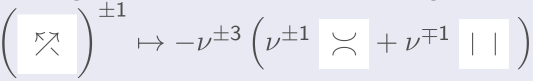
But: Above assignment is **not** invariant under the Markov move!

Recall $\delta = -\nu^2 - \nu^{-2}$:





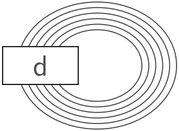
We obtain an assignment invariant under the Markov move by passing to **oriented** links and letting



Now both  and  are mapped to | .

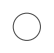
Link invariants from the Temperley–Lieb algebra

We define the trace $\mathsf{Tr} : \mathsf{TL}_n(\delta) \to \mathbb{C}$ by "closing the diagram"

$\mathsf{Tr}(d) =$ 

, where each circle gets evaluated to δ.

Let $q := -\nu^{-2}$. Recall $\delta = -\nu^2 - \nu^{-2} = q + q^{-1}$.

The Markov invariant assignment together with the trace map define an invariant J for oriented links with normalization $J($  $) = \delta = q + q^{-1}$

and skein relation $q^2 J($  $) - q^{-2} J($  $) = (q - q^{-1}) J(\uparrow\uparrow)$.

This is the Jones polynomial (up to the normalization)!

Gimme s'more!

There is a family of invariants $(P_n)_{n \geq 0}$ with skein relation

$q^n P_n($  $) - q^{-n} P_n($  $) = (q - q^{-1}) P_n(\uparrow\uparrow)$

and the normalization $P_n(\bigcirc) = \frac{q^n - q^{-n}}{q - q^{-1}}$.

$P_0 =$ Alexander polynomial, $P_1 \equiv 1$, $P_2 =$ Jones polynomial, ...

- All of these can be obtained from quantum groups, too.
- The HOMFLYPT polynomial is a 2-parameter generalization.
- The HOMFLYPT polynomial is not a complete invariant.
- Categorification $\Rightarrow$ HOMFLYPT is the Euler characteristic of "Khovanov's triply graded link homology".

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