Introduction to Quantum Groups and Tensor Categories

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[1] Got questions or comments? Just get in touch with him.

Outline

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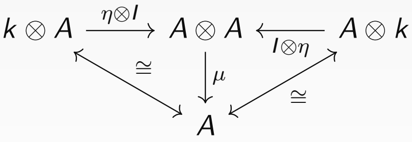
"A mathematican is a machine for turning coffee into theorems."
Alfréd Rényi (often attributed to Paul Erdős)

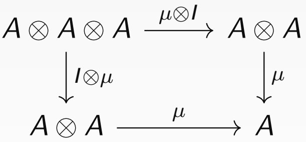
"A comathematican is a machine for turning cotheorems into ffee." communicated to the author by Fei Qi

(Co-)Algebras

k: our favorite commutative r1ng/field, all maps are k-linear.

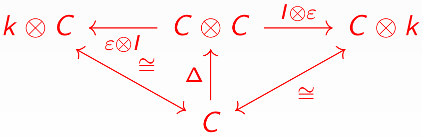
- *Algebra*: k-space A with $\eta : k \to A$, $\mu : A \otimes A \to A$
- *Coalgebra*: k-space C with $\varepsilon : C \to k$, $\Delta : C \to C \otimes C$

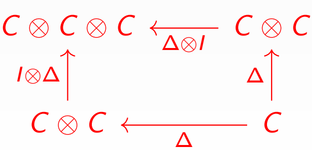




(co-)unitarity

(co-)associativity





Convolution

- *Sweedler's Notation:* $\forall x \in C$

$$\Delta(x) = \sum_{i=1}^{n} x_{1,i} \otimes x_{2,i} =: x_1 \otimes x_2 \in C \otimes C$$

  coassociativity $\Rightarrow$ "$x_1 \otimes x_2 \otimes x_3$" is well-defined
  counitarity $\Leftrightarrow$ $\varepsilon(x_1)x_2 = x = x_1\varepsilon(x_2)$

- *Convolution:* $\forall f, g : C \to A$, $f * g := \mu \circ (f \otimes g) \circ \Delta$,
  i.e. $(f * g)(x) := f(x_1)g(x_2)$ $\forall x \in C$
- Note that $\eta \circ \varepsilon : C \to A$ is an identity element for $*$:
  $\forall f : C \to A, x \in C$,

$$(f * (\eta \circ \varepsilon))(x) = f(x_1)\eta(\varepsilon(x_2)) = f(x_1\varepsilon(x_2))1 = f(x)$$

$$((\eta \circ \varepsilon) * f)(x) = f(\varepsilon(x_1)x_2) = f(x)$$

Bialgebras, Hopf algebras

- *Bialgebra*: algebra and coalgebra with compatible structure maps (η, μ are coalgebra maps, ε, Δ are algebra maps.)
- *Hopf algebra:* bialgebra H with an antipode, that is a $*$-inverse S of I as maps $H \to H$. For all $x \in H$, this means

$$x_1 S(x_2) = (I * S)(x) = \varepsilon(x)1 = (S * I)(x) = S(x_1)x_2$$

$\Rightarrow$ S is an antialgebra map and an anticoalgebra map, every bialgebra has at most one antipode.

Examples

- *Group algebra* $k[G]$ for a group G
  basis: $\{g\}$ for $g \in G$
  $\varepsilon g = 1$, $\Delta g = g \otimes g$, $Sg = g^{-1}$
  ("*group-like element*")
- *Universal enveloping algebra* $U(\mathfrak{g})$ for a Lie group $\mathfrak{g}$
  basis: $\{x_1^{p_1} \cdots x_n^{p_n} | p_1, \ldots, p_n \geq 0\}$ for a basis $x_1, \ldots, x_n$ of $\mathfrak{g}$
  $\varepsilon x_i = 0$, $\Delta x_i = 1 \otimes x_i + x_i \otimes 1$, $Sx_i = -x_i$
  ("*primitive element*")

$\Rightarrow$ In both cases, $S^2 = I$.
Any cocommutative Hopf algebra over $\mathbb{C}$ is generated by group-likes and primitives.[2]

[2] Any cocommutative Hopf algebra over $\mathbb{C}$ is the *semidirect/smash product Hopf algebra* of the group algebra of the group formed by its group-likes and the universal enveloping algebra of the Lie algebra formed by its primitives.

Categories and their bialgebras

"Tannaka(-Krein) duality", "reconstruction theorems"

- Rep(A): category of modules of an algebra A of finite rank/dimension over k
- Consider categories "of k-modules of finite rank/dimension".

category	Rep(...)
vector spaces/modules	k
monoidal	bialgebra
rigid monoidal	Hopf algebra
rigid braided monoidal	quasitriangular Hopf algebra
Ribbon	Ribbon Hopf algebra

Tannaka-Krein duality

- "For A an algebra and AMod its category of modules, and for AMod $\to$ Vect the fiber functor that sends a module to its underlying vector space, we have a natural isomorphism End(AMod $\to$ Vect) $\simeq A$ in Vect." [3]
- "The assignments

$$(\mathcal{C}, F) \mapsto H = \mathrm{End}(F), H \mapsto (\mathrm{Rep}(H), \mathrm{Forget})$$

are mutually inverse bijections between (1) equivalence classes of finite tensor categories $\mathcal{C}$ with a fiber functor F, up to tensor equivalence and isomorphism of tensor functors, and (2) isomorphism classes of finite dimensional Hopf algebras over k." [4]

[3] https://ncatlab.org/nlab/show/Tannaka+duality
[4] thm. 5.3.12 in Etingof, Gelaki, Nikshych, Ostrik: *Tensor Categories*.

Outline

R-matrices

We fix a Hopf algebra A over k.

- $\forall V, W$ k-spaces, $\tau_{V,W} : V \otimes W \to W \otimes V, v \otimes w \mapsto w \otimes v$.
- $\forall R \in A^{\otimes 2}$ we define elements in $A^{\otimes 3}$:
  $R_{12} := R \otimes 1$, $R_{23} := 1 \otimes R$, $R_{13} := (I \otimes \tau)(R \otimes 1)$.

$R \in A^{\otimes 2}$ is called *(universal) R-matrix*, if

1. R is invertible and $\tau \circ \Delta(a) = R\Delta(a)R^{-1}$
2. $(I \otimes \Delta)R = R_{13}R_{12}$
3. $(\Delta \otimes I)R = R_{13}R_{23}$

$\Rightarrow (\varepsilon \otimes I)R = (I \otimes \varepsilon)R = 1 \otimes 1$, $(S \otimes I)R = (I \otimes S^{-1})R = R^{-1}$

$\Rightarrow R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$ "*Yang-Baxter Equation*"

Scribble (some proofs)

$R =: R^1 \otimes R^2 =: r^1 \otimes r^2 \in A^{\otimes 2}$, summation implied (but not a coproduct!).

$$(\Delta \otimes I)R = R_{13}R_{23} \Leftrightarrow R_1^1 \otimes R_2^1 \otimes R^2 = r^1 \otimes R^1 \otimes r^2 R^2 \ldots$$

$$\begin{aligned} \ldots \Rightarrow \varepsilon(R_1^1) \otimes R_2^1 \otimes R^2 &= \varepsilon(r^1) \otimes R^1 \otimes r^2 R^2 \\ \Rightarrow 1 \otimes R^1 \otimes R^2 &= 1 \otimes \varepsilon(r^1) R^1 \otimes r^2 R^2 \\ \Rightarrow 1 \otimes 1 &= \varepsilon(r^1) \otimes r^2 \end{aligned}$$

$$\begin{aligned} \ldots \Rightarrow S(R_1^1) R_2^1 \otimes R^2 &= S(r^1) R^1 \otimes r^2 R^2 \\ \Rightarrow \varepsilon(R^1) \otimes R^2 &= (S(r^1) \otimes r^2)(R^1 \otimes R^2) \\ \Rightarrow 1 \otimes 1 &= (S(r^1) \otimes r^2) R \end{aligned}$$

Representations of quasitriangular Hopf algebras

If A has an R-matrix R, it is called *quasitriangular*.
In this case, we define maps for all pairs of objects $V, W \in \mathrm{Rep}(A)$:

$$c_{V,W} : V \otimes W \to W \otimes V, x \mapsto \tau(Rx) \ .$$

$\Rightarrow$ Then $\mathrm{Rep}(A)$ is a braided monoidal category with braiding c, i.e. for any $n \geq 1$, the braid group B_n acts on n-fold tensor products of A-modules via c.

$u := \mu \circ (S \otimes I) \circ \tau(R) \in A$
$\Rightarrow$ u is invertible and $S^2(a) = uau^{-1}$, $\forall a \in A$
(compare this with our examples for Hopf algebras above)

$\Rightarrow u^{-1} = (I \otimes S^2)\tau(R), \varepsilon(u) = 1, \Delta u = (\tau(R)R)^{-1}(u \otimes u)$

Ribbon elements

We fix a quasitriangular Hopf algebra A with R-matrix R.

A central invertible $v \in A$ is called *universal twist* or *ribbon element* if

1. $v^2 = uS(u)$
2. $\varepsilon(v) = 1$
3. $\Delta v = (\tau(R)R)^{-1}(v \otimes v)$
4. $S(v) = v$

Note: If $v = ug^{-1}$ for a group-like g, then (2), (3) follow directly and (1), (4) are equivalent.

Representations of ribbon Hopf algebras

If A has a Ribbon element v, it is called *ribbon Hopf algebra*.
In this case, we define maps for all objects $V \in \mathrm{Rep}(A)$:

$$\theta_V : V \to V, x \mapsto vx \ .$$

$\Rightarrow$ Then $\mathrm{Rep}(A)$ is a Ribbon category with twist θ, i.e. $\forall V, W$,

- $\theta_{V \otimes W} = c_{W,V} c_{V,W} (\theta_V \otimes \theta_W)$
- $(\theta_V \otimes I_{V^*}) b_V = (I_V \otimes \theta_{V^*}) b_V$, where $b_V : k \to V \otimes V^*$.

Outline

Definition

- *quantum group* $\overset{\text{here}}{:=}$ *quantized universal enveloping algebra*
- $(a_{ij})_{1\le i,j\le m}$ the Cartan matrix of a simple Lie algebra $\mathfrak{g}$ of type ADE ($\Rightarrow a_{ii}=2$, $a_{ij}=a_{ji}\in\{0,-1\}$ for $i\neq j$)
- $q\in\mathbb{C}\setminus\{0,\pm1\}$

$U_q(\mathfrak{g})$ generated by $\{E_i,F_i,K_i,K_i^{-1}\}_{1\le i\le m}$ with relations:

$$[K_i,K_j]=0 \qquad K_iK_i^{-1}=1=K_i^{-1}K_i$$

$$K_iE_j=q^{a_{ij}}E_jK_i \qquad K_iF_j=q^{-a_{ij}}F_jK_i \qquad [E_i,F_j]=\delta_{ij}\frac{K_i-K_i^{-1}}{q-q^{-1}}$$

$$[E_i,E_j]=[F_i,F_j]=0 \qquad \text{if } a_{ij}=0$$

$$\left.\begin{array}{l} E_i^2E_j-(q+q^{-1})E_iE_jE_i+E_jE_i^2=0 \\ F_i^2F_j-(q+q^{-1})F_iF_jF_i+F_jF_i^2=0 \end{array}\right\} \qquad \text{if } a_{ij}=-1$$

Definition/Theorem

$U_q(\mathfrak{g})$ is a Hopf algebra with

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i \qquad S(E_i) = -K_i^{-1} E_i \qquad \varepsilon(E_i) = 0 \ ,$$
$$\Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i \qquad S(F_i) = -F_i K_i \qquad \varepsilon(F_i) = 0 \ ,$$
$$\Delta(K_i) = K_i \otimes K_i \qquad S(K_i) = K_i^{-1} \qquad \varepsilon(K_i) = 1 \ .$$

Assume q is a p-th root of unity, $p \geq 3$, $p' := \begin{cases} p & p \text{ odd} \\ p/2 & p \text{ even} \end{cases}.$

$J := \langle E_i^{p'}, F_i^{p'}, K_i^p - 1 \rangle_i$ as ideal in $U_q(\mathfrak{g})$.

$\Rightarrow \tilde{U}_q(\mathfrak{g}) := U_q(\mathfrak{g})/J$ is a fin.-dim. ribbon quotient Hopf algebra.

Scribble (proof ideas)

- We may verify that $U_q(\mathfrak{g})$ is a Hopf algebra, and that J is a Hopf ideal. Hence $\tilde{U}_q(\mathfrak{g})$ is a Hopf algebra.
- It is quasitriangular, because it is the quotient of a Drinfel'd double (see following slides).

- Let $(b_{ij})_{i,j} := (a_{ij})_{i,j}^{-1}$, $b_i := \sum_j b_{ij}$, $g := K_1^{-2b_1} \cdots K_m^{-2b_m}$.
  $\Rightarrow$ g is an invertible group-like in $\tilde{U}_q(\mathfrak{g})$,
  $S^2(a) = gag^{-1}$ for all $a \in \tilde{U}_q(\mathfrak{g})$
- Let u be the distinguished element of the quasitriangular Hopf algebra $\tilde{U}_q(\mathfrak{g})$.
  $\Rightarrow$ $v := ug^{-1}$ is central invertible and we may also verify that $Sv = v$. Hence, v is a ribbon element.

Drinfel'd double

Consider

- A a fin.-dim. Hopf algebra with dual A^*
- $A^0 := A^*$ as algebra, but with $\Delta^0 := \tau \circ \Delta$, $S^0 := S^{-1}$

$\Rightarrow \exists$ Hopf algebra $D(A) \simeq A \otimes A^0$ as k-spaces such that the identifications $A \to A \otimes 1 \subset D(A)$ and $A^0 \to 1 \otimes A^0 \subset D(A)$, are Hopf algebra maps and such that their images generate $D(A)$ as algebra.

$D(A)$ is quasitriangular, with R the identity element in $A \otimes A^0$ (A has to be finite-dimensional!).

Note: $D(A)$ can be defined even if A is not finite-dimensional, and even for two Hopf algebras with a suitable pairing.

Note also: $D(A)$ is the Hopf algebra corresponding to the *"center"* of the tensor category Mod(A) by Tannaka-Krein duality.

Yetter-Drinfel'd modules, Radford's biproduct/bosonization

For a Hopf algebra H, ${}^H_H\mathrm{YD}$ is the category of (left left) (H,H)-bimodules V with compatibility condition

$$\delta(h.v) = h_1 v_{-1} S h_3 \otimes h_2.v_0 \qquad \forall h \in H, v \in V,$$

where δ is the coaction and $\delta(v) =: v_{-1} \otimes v_0$.
$\Rightarrow {}^H_H\mathrm{YD}$ is a braided monoidal category

$\exists$ functor *Radford's biproduct/bosonization*
$\{$"braided" Hopf algebra in ${}^H_H\mathrm{YD}\} \to \{$Hopf algebra$\}, A \mapsto A\#H$.[5]
$A\#H$ contains H as Hopf subalgebra and A as subalgebra.

[5] Not to be confused with the *semidirect/smash product* which is sometimes denoted identically. The latter one is a product of a Hopf algebra and a module algebra, and no comodule structure is involved.

Quantum groups revisited

$H := k[\mathbb{Z}^m] = k[K_1, \dots, K_m]$, $V^\pm := k^n = \oplus_{i=1}^m E_i^\pm k$ the Yetter-Drinfel'd modules defined by $K_i.E_j^\pm = q^{\pm a_{ij}}$ and $\delta(E_i^\pm) = K_i \otimes E_i^\pm$.

- $T(V^\pm)$ are braided Hopf algebras
- adding the *Serre relations* $\mathrm{ad}_{E_i^\pm}^{1-a_{ij}}(E_j^\pm) = 0$ to $T(V^\pm)$
  $\rightarrow$ braided Hopf algebras $U(\mathfrak{n}^\pm)$
  ("*Borel part*"; ad is to be taken in ${}^H_H\mathrm{YD}$)
- bosonizations $U(\mathfrak{n}^\pm)\#H$
  $\rightarrow$ Hopf algebras which are dual in the sense of $A \mapsto A^0$
- $U_q(\mathfrak{g})$: Drinfel'd double $D(U(\mathfrak{n}^+)\#H)$ modulo identification of the two copies of H. $E_i = E_i^+$, $F_i = E_i^-$.

Quantum groups revisited / Outlook

- Drinfel'd doubles and quotients of quasitriangular Hopf algebras are quasitriangular, so $\tilde{U}_q(\mathfrak{g})$ is quasitriangular
- Generalizations of the quantum groups discussed here which are still Ribbon Hopf algebras have been defined[6].
  The fact that quantum groups and their generalizations are ribbon Hopf algebras can be proved through general Hopf algebra theory, as well[7].
- There are results on how braided tensor categories obtained from conformal field theories can be studied through quantum groups[8].

[6] Majid, *Double-bosonization of braided groups and the construction of* $U_q(\mathfrak{g})$, 1996 / Heckenberger, *Nichols Algebras (Lecture Notes)*, 2008 / ...
[7] Burciu, *A class of Drinfeld doubles that are ribbon algebras*, 2008.
[8] see `http://arxiv.org/pdf/0705.4267v2.pdf`, for instance

Summary

category	Rep(...)
vector spaces / modules	k
monoidal	bialgebra
rigid monoidal	Hopf algebra
rigid braided monoidal	quasitriangular Hopf algebra
Ribbon	Ribbon Hopf algebra*

(*) e.g. quantum groups

Quantum groups are quotients of Drinfel'd doubles of bosonizations of universal enveloping algebras of Borel subalgebras of Lie algebras in a category of Yetter-Drinfel'd modules. Roughly speaking.

For further reading

- Turaev, *Quantum Invariants of Knots and 3-Manifolds*, 1994: chapter XI 1-3, 6.
- Chari, Pressley, *A Guide to Quantum Groups*, 1995.
- Majid, *Foundations of Quantum Group Theory*, 2000.
- Drinfel'd, *Quantum Groups*, 1986, [here].
- Reshetikhin, Turaev, *Invariants of 3-manifolds via link polynomials and quantum groups*, 1991, [here].
- Heckenberger, *Nichols Algebras (Lecture Notes)*, 2008, [here]: section 7, see also Simon Lentner's MO answer [here].